Regularization vs Step-Size Regulators in Optimal Trajectory Computations

Andrew J. Czuchry* and Edward T. Pitkin†
The University of Connecticut, Storrs, Conn.

Introduction

It has been demonstrated[1,2] that the use of regularized variables increases accuracy and reduces integration time for the computation of coasting space trajectories. The need for improved integration methods is even greater for optimal powered trajectories because of the introduction of additional equations describing the necessary conditions for optimality and because a two-point boundary value problem usually arises requiring some means of iterative solution involving multiple integrations of the differential equations. It is natural then to examine possible benefits of regularization for such problems. Several authors have recently applied regularizing techniques to the minimum time, powered transfer problem. Reference 3 indicated much success when using regularized variables for these optimal trajectory computations, while in Ref. 4 it was noted that computational advantages can be realized by a judicious selection of a coordinate system.

The purposes of this Note are to examine the regularization procedure in order to determine what it contributes numerically to the problem solution and to offer an alternative means for achieving improved numerical accuracy.

Optimization Problem

We consider planar, constant-power rocket trajectories optimized for minimum propellant consumption. Sample trajectories that pass close to the attracting mass centers for both the two-body and two-fixed-center problems are considered. The latter presents computational difficulties of the type encountered in the restricted three-body problem. The differential equations for the perturbed (by rocket thrust) two-fixed-center problem are

$$\begin{aligned} \ddot{x}_1 &= -\mu\xi/r_2^3 - \psi\zeta/r_1^3 + \lambda_1 \\ \ddot{x}_2 &= -\mu x_2/r_2^3 - \psi x_2/r_1^3 + \lambda_2 \end{aligned} \tag{1}$$

where

$$\xi = x_1 + 1 - \mu,\ \psi = 1 - \mu,$$
$$\zeta = x_1 - \mu,\ \mu = m_2/(m_1 + m_2)$$

and

$$r_1 = (\zeta^2 + x_2^2)^{1/2},\ r_2 = (\xi^2 + x_2^2)^{1/2}$$

The dot implies differentiation with respect to the physical time, t. The origin is located at the center of mass and the sum of the masses is 1. The coordinate system geometry for the case $\mu = \frac{1}{2}$ is given in Fig. 1. Equations (2) are the adjoint or Euler-Lagrange equations which arise from the necessary conditions for optimality. In the power limited case the adjoint variables λ_1 and λ_2 also become the components of the thrust acceleration vector.

$$\begin{aligned} \ddot{\lambda}_1 &= \lambda_1\left[\frac{-\mu}{r_2^3} + \frac{3\mu\xi^2}{r_2^5} - \frac{\psi}{r_1^3} + \frac{3\psi\zeta^2}{r_1^5}\right] + \lambda_2\left[\frac{3\mu x_2\xi}{r_2^5} + \frac{3\psi x_2\zeta}{r_1^5}\right] \\ \ddot{\lambda}_2 &= \lambda_1\left[\frac{3\mu\xi x_2}{r_2^5} + \frac{3\psi\zeta x_2}{r_1^5}\right] + \lambda_2\left[\frac{-\mu}{r_2^3} + \frac{3\mu x_2^2}{r_2^5} - \frac{\psi}{r_1^3} + \frac{3\psi x_2^2}{r_1^5}\right] \end{aligned} \tag{2}$$

If μ is taken equal to 0 in Eqs. (1) and (2), the attracting mass m_2 on the left in Fig. 1 disappears and the center of the coordinate system shifts to the remaining attracting mass m_1. The reduced set of equations, with r_1 becoming r, then characterize the motion of a rocket in the presence of a single attracting center, i.e., the perturbed two-body problem.

In the examples considered here, these equations are to be solved subject to boundary conditions which are given as the initial and final values of the state vector $(x_1,x_2,\dot{x}_1,\dot{x}_2)$. The final value of the physical time, t_f, is also given.

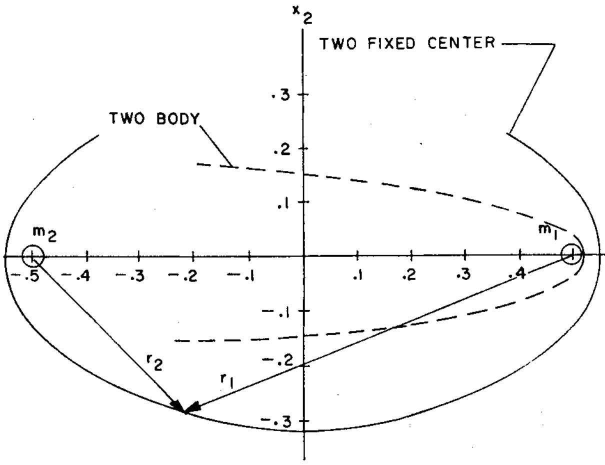

Fig. 1 Perturbed two-fixed-center and perturbed two-body optimal low-thrust trajectories.

Regularization

Regularization of the unthrusted two-body problem can be accomplished with the aid of the Sundman time transformation,[3] $dt = rd\tau$, where τ is called the pseudo time. The effect of transforming to the τ domain is to expand the pseudo time interval in the vicinity of the singularity, $r = 0$, so that the transformed velocity, $dr/d\tau$, remains well behaved if the two bodies collide. In the unthrusted two-fixed-center problem it is desirable to have a transformation that simultaneously removes the computational difficulties which occur when the rocket passes by either attracting center. The Thiele transformation, $dt = r_1r_2d\tau$, will serve this purpose adequately.

Regularization can be applied to optimization problems either by first regularizing the equations of motion and then setting down the necessary criteria for optimization in terms of the pseudo-time or by first setting up the optimization problem in physical time and applying the regularizing transformation to the complete set of equations.[5] In either event, the basic Sundman and Thiele time transformations should be modified because the adjoint equations exhibit singularities of higher order than the state equations. Furthermore, the partial derivatives required to form the variational equations used in the solution of the resulting two-point boundary value problem possess even higher order singularities. Therefore, the time transformations which will be considered here are the generalized Sundman transformation $dt = r^n d\tau$ for

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* NDEA and University Predoctoral Fellow; now with Dynamics Research Corporation.
† Professor of Aerospace Engineering.

the perturbed two-body problem and the generalized Thiele transformation $dt = r_1^n r_2^m d\tau$ for the perturbed two-fixed-center problem.

The values of n and m should be chosen large enough so that there will be no troublesome terms in the transformed equations. It might appear that they should be taken large enough to regularize the highest-order singularity in the system of equations. However, if n and m are chosen larger than $\frac{3}{2}$, the final value of the pseudo time will become large without bound when either r_1 or r_2 goes to zero. The practical implication is that the pseudo time may be over expanded in those portions of the trajectory that pass close to the attracting force centers if n and m are taken much larger than $\frac{3}{2}$. An unduly large number of integration steps may then be required to traverse this region. Full regularization of the system of optimization equations is then neither desirable or necessary as long as actual collisions are avoided. It appears reasonable then to select the order of the regularizing function at least as large as the value required to regularize the state and adjoint equations along a coast trajectory.

With a value of $\frac{3}{2}$ for n and m the state and adjoint Eqs. (1) and (2) transform to Eqs. (3) and (4), respectively. In the powered case velocity terms of the state equations are regular if the thrust remains bounded near collision[6] and it is apparent that the powers of the denominator terms of the adjoint equations are now considerably reduced.

$$x_1'' = 3r_1'x_1'/2r_1 + 3r_2'x_1'/2r_2 - \mu\xi r_1^3 - \psi\zeta r_2^3 + \lambda_1 r_2^3 r_1^3 \quad (3a)$$

$$x_2'' = 3r_1'x_2'/2r_1 + 3r_2'x_2'/2r_2 - \mu x_2 r_1^3 - \psi x_2 r_2^3 + \lambda_2 r_1^3 r_2^3 \quad (3b)$$

$$\lambda_1'' = \frac{3r_1'\lambda_1'}{2r_1} + \frac{3r_2'\lambda_1'}{2r_2} + \lambda_1\left[-\mu r_1^3 + \frac{3\mu\xi^2 r_1^3}{r_2^2} - \psi r_2^3 + \frac{3\psi\zeta^2 r_2^3}{r_1^2}\right] + \lambda_2\left[\frac{3\mu x_2\xi r_1^3}{r_2^2} + \frac{3\psi\zeta r_2^3 x_2}{r_1^2}\right] \quad (4a)$$

$$\lambda_2'' = \frac{3r_1'\lambda_2'}{r_1} + \frac{3r_2'\lambda_2'}{r_2} + \lambda_1\left[\frac{3\mu x_2\xi r_1^3}{r_2^2} + \frac{3\psi x_2\zeta r_2^3}{r_1^2}\right] + \lambda_2\left[-\mu r_1^3 + \frac{3\mu x_2^2 r_1^3}{r_2^2} - \psi r_2^3 + \frac{3\zeta x_2^2 r_2^3}{r_1^2}\right] \quad (4b)$$

In these equations the prime implies differentiation with respect to the pseudo time, τ. It is also apparent that these equations are more complex than their untransformed counterparts as they contain more terms as well as more products in the individual terms. This effect is increased when the variations of these equations are computed for solution of the two-point boundary value problem. For instance in the two-fixed-center problem it requires about three times as much computer coding for the transformed system of equations as it does for the standard form.

Another unfavorable aspect should also be considered. In the transformed time domain, the state vector must now be augmented by the time t which takes on the role of an additional state variable governed by the differential equation of the transformation

$$dt/d\tau = r_1^{3/2} r_2^{3/2} \quad (5)$$

It should be noted from integration of this equation that τ_f cannot be specified even though t_f is given, because the value of τ_f depends upon the path. Thus, either a fixed or open final time problem in t must become an open final time problem in τ[5]. Since the original problem was autonomous in the physical time, the additional adjoint equation simply becomes $\lambda_t = \text{const.}$

A way of avoiding some of these unfavorable aspects is suggested if one considers what regularization really accomplishes from a computational point of view. From this standpoint the regularity of the transformed equations implies an automatic integration step size regulator, i.e., constant integration step size, $\Delta\tau$, in the regularized time domain is equivalent to variable integration step size in physical time, t. Thus, an alternative means for achieving improved numerical accuracy is suggested: integration of the untransformed equations in physical time with the integration step size selected in accordance with the regularizing function, i.e.,

$$\Delta\tau \sim r_1^n r_2^m \quad (6)$$

If the resultant accuracy of the two approaches is equivalent, this alternative offers increased computational speed and simplicity. Of course the payment for this advantage is the loss of the implicit step size regulator. This cost may be considerable when high-order multistep integration algorithms are used because they do not lend themselves to step size adjustment at every integration step. However, one might find step size adjustment on a less frequent basis to be practical with multistep algorithms.

Both the "regularization" and "step-size regulator" approaches will be compared in the next section.

Numerical Results

Two different low-thrust trajectories and a coast trajectory were integrated to determine the effects of regularization in optimal trajectory computations. The numerical integrations were all performed in double precision with a one-step, fourth-order, Runge-Kutta algorithm. This algorithm is not noted for high accuracy but is suitable for tests of relative accuracy as long as the same algorithm is used for all cases. It was selected because it can easily be coded for variable integration step size.

The first example considered here is a minimum propellant consumption, low thrust trajectory that passes close to a single attracting center m_1. This was obtained by perturbing the end state and final time for a symmetrical two-body coast arc that passed close to m_1, thus we have a perturbed two-body problem. The resulting trajectory is

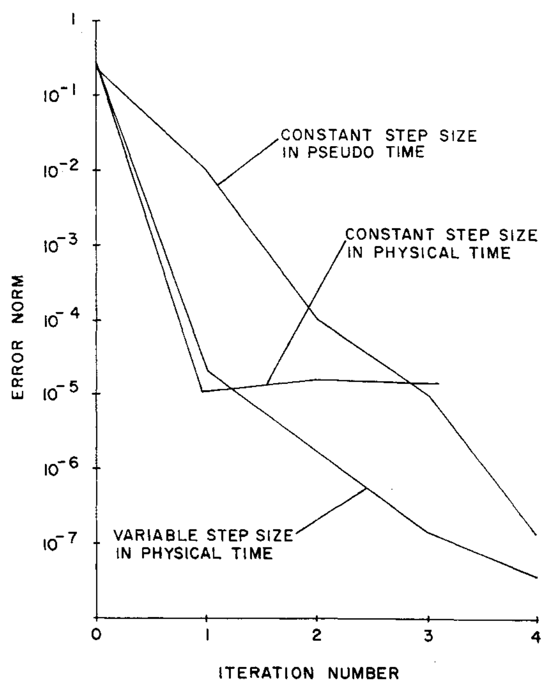

Fig. 2 Comparison of error norms for perturbed two-body problem.

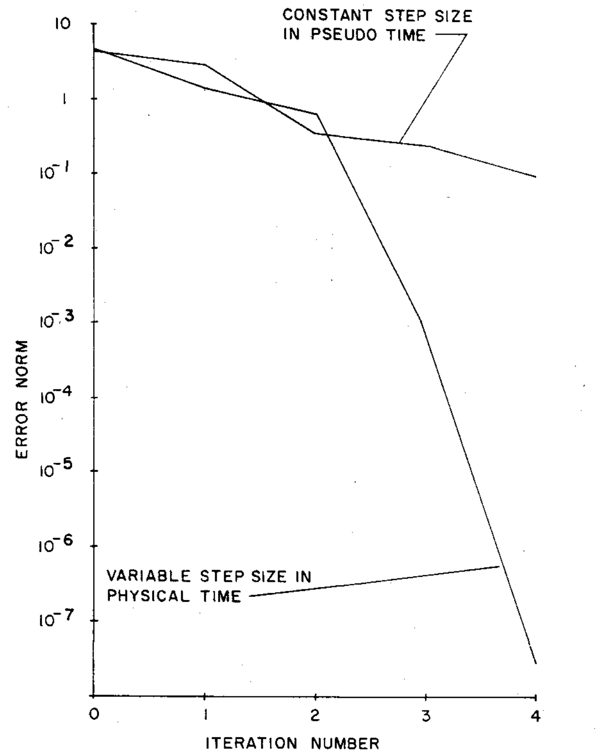

Fig. 3 Comparison of error norms for perturbed two-fixed-center problem.

sketched as the dashed curve in Fig. 1. The basic equations for this problem in physical time can be obtained from Eqs. (1) and (2) by setting $\mu = 0$. The transformed set of equations can then be obtained via the generalized Sundman transformation $dt = r^{3/2}d\tau$.

Three different ways of obtaining numerical solutions were compared: a constant integration step size in physical time (method I); a physical time solution for which the integration step size was selected in accordance with the regularizing function (method II); and a regularized time domain solution (method III). In each case the appropriate equations, Eqs. (1) and (2) or (3) and (4) were converted to first-order form and solved subject to the prescribed boundary conditions by the modified Newton-Raphson process described in Ref. 7. In this method the unknown initial conditions are modified at each iteration until they finally converge to their final and hopefully correct values. The "correct" values can be obtained by solving the problem with an exceedingly small integration step size. When the same answer is obtained by methods II and III, one may be reasonably assured of the "correctness" of the answer. When using larger step sizes, the difference between the last value of the unknown initial conditions and the "correct" values may be taken as a measure of the error at each iteration. The error norm used here is the sum of the absolute values of these differences.

Starting with estimates of the unknown initial conditions (the adjoint variables in this case) 100% in error and the final value of the pseudo time 10% in error in method III, the optimal trajectory was obtained by all three methods. The results are summarized in Fig. 2 where the error norms are plotted against iteration number. Here 1500 integration steps were used for method I and only 400 steps were used for methods II and III. The constant step size, physical time solution shows no improvement after the first iteration. Both the regulated step size, physical time solution, and regularized variable solution continued to improve through the fourth iteration and are then competitive in accuracy. It should be recalled however that the unregularized equations are easier to code and require less computing time.

The next example, based upon the perturbed two-fixed-center problem, is the low thrust minimum propellant trajectory illustrated in Fig. 1 by the solid line. Again the test trajectory was obtained by perturbing the end states and final time of a coast trajectory. The correct solution was first obtained again by using very small step sizes in methods II and III.

The first observation from this study is that the numerical difficulties experienced when the rocket passes by two singularities are more pronounced than for the case when only one singularity is present. In fact it was found that computation time became prohibitive to obtain more than five significant figure accuracy with a constant integration step size in physical time, method I. The need to convert to an open final time problem when using the regularization approach caused additional difficulties in converging to the proper end states. In this case a low-amplitude oscillation about the correct value of t_f was experienced so that the error norm remained quite large as shown in Fig. 3. The regulated step size approach led to no such difficulty however, and convergence to an error norm of 10^{-7} was readily obtained in four iterations. It would appear then that there can be a definite disadvantage to using regularization if one is dealing with a fixed final time problem.

A higher-order regularizing transformation, $dt = r_1{}^3r_2{}^3d\tau$ was also tried to see if it gave any numerical improvement. No significant differences were observed, however.

For a final related experiment the state equations, Eqs. (1) or (3), were integrated by all three methods for a two-fixed-center coast trajectory similar in shape to that shown in Fig. 1. In each case 960 integration steps were used. Method I, the constant step size in physical time, gave four significant figure accuracy, while methods II and III, the regulated step size in physical time and constant step size in pseudo-time respectively, both gave eight significant figures in the end states. Here, of course, the final value of pseudo time τ_f could be determined accurately because the trajectory was known a priori. It would appear then that a step size in physical time regulated according to the criterion $\Delta t \sim r_1{}^n r_2{}^m$ offers the same improvement in accuracy as does the regularization approach.

Conclusions

The preceding experiments suggest very strongly that the major effect of regularization by time transformations is to implement an implicit integration step size regulator which significantly improves accuracy. This could be advantageous when dealing with high-order multistep algorithms with which it is difficult to change step size readily. On the the other hand, the time transformation introduces much additional complexity to the equations of motion. Furthermore, it complicates the two-point boundary value problem by converting a fixed final time problem to an open final time problem.

The variable integration step size in physical time, or step-size regulator approach, offers the same improvement in accuracy with only a single disadvantage of requiring a step size change at every step. In this form it could only be considered for use with one-step algorithms; however, one could also apply the same idea to multistep algorithms by changing step size periodically according to the criterion given here. Furthermore, the step size in physical time may be adjusted according to any rational criterion that decreases Δt in the region where r_1 or r_2 is small without regard to it being a proper regularizing transformation. Significant improvement in computational accuracy should then be anticipated.

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Some Simple Scaling Relations for Heating of Ballistic Entry Bodies

MICHAEL E. TAUBER*
NASA Ames Research Center, Moffett Field, Calif.

Nomenclature

A = base area of entry body
B = ballistic parameter, see Eq. (3)
C = constant, see Eq. (4)
C_D = drag coefficient
k = constant equal to $B/2(M-1)$
m = body mass
M = constant, see Eq. (4)
N = constant, see Eq. (4)
q = heat transfer into the body per unit area
Re = Reynolds number
t = time
V = flight velocity
y = flight altitude
β = inverse scale height of atmosphere
γ = flight-path angle below horizontal
ρ = freestream density

Subscripts

E = entry
F = final
0 = reference value (for Earth — sea level)

BALLISTIC atmospheric entries can be characterized by three parameters. These describe the body, the flight path, and the atmosphere; they are the ballistic coefficient m/C_DA, the flight-path angle γ, and the inverse atmospheric scale height β, respectively. In this Note some scaling relations are derived to show the influence of the aforementioned three parameters on the total heating, per unit area, of a body making a ballistic entry.

During that part of the entry where aerodynamic heating is important, the drag is generally much greater than the weight component along the flight path and the equation of motion can be written

$$\tfrac{1}{2}\rho V^2 C_D A \doteq -m(dV/dt) \tag{1}$$

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* Research Scientist. Associate Fellow AIAA.

If the flight-path angle is not too shallow ($\gamma_E \geq 8°$, approximately, at the beginning of the sensible atmosphere), it can be assumed that

$$\sin\gamma \doteq \text{const} \doteq \sin\gamma_E \tag{2}$$

Using the assumption of an exponential atmospheric density variation with altitude, Allen and Eggers[1] integrated the equation of motion and showed that

$$V/V_E = e^{-(B/2)\bar{\rho}} \tag{3}$$

where

$$B = (C_DA/m)(\rho_0/\beta \sin\gamma_E) = \text{const}$$

and

$$\bar{\rho} = \rho/\rho_0$$

Now, it is assumed that the heating rate per unit area can be written in the form

$$(dq/dt) = C(\bar{\rho})^N V^M \tag{4}$$

where N, M, and C are constants. Equation (4) is a reasonable approximation for both laminar[2] ($N = 0.5$, $M \doteq 3.1$) and turbulent[3] ($N \doteq 0.8$, $M \doteq 3.7$) convection if ablation product blockage is negligible. For radiation, Eq. (4) is only a crude approximation; for continuum air radiation $N = 1.8$ and $M = 15.5$ (for $V < 13.7$ km/sec) has been used.[3]

The total heating, per unit area, experienced by the body during the entry is

$$q = \int_0^t \frac{dq}{dt}\,dt \tag{5}$$

By combining Eqs. (1, 3, and 4) and substituting the result into Eq. (5)

$$q = \frac{CV_E^{M-1}}{\beta \sin\gamma_E}\int_0^1 (\bar{\rho})^{N-1}\, e^{-(B/2)(M-1)\bar{\rho}} d\bar{\rho} \tag{6}$$

Setting

$$k = (B/2)(M-1)$$

and rewriting the integral

$$\int_0^1 (\bar{\rho})^{N-1} e^{-k\bar{\rho}}\, d\bar{\rho} = \int_0^\infty (\bar{\rho})^{N-1} e^{-k\bar{\rho}}\, d\bar{\rho} - \int_1^\infty (\bar{\rho})^{N-1} e^{-k\bar{\rho}}\, d\bar{\rho}$$

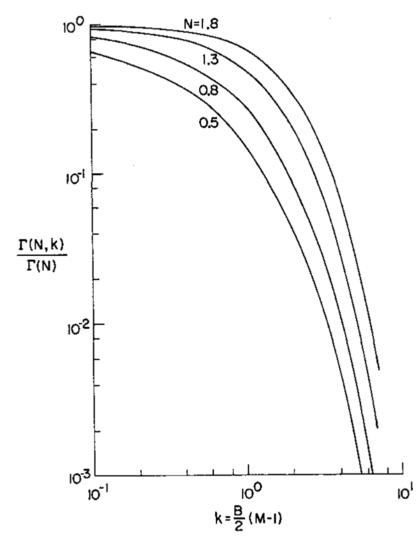

Fig. 1 Ratio of incomplete to complete gamma function.